

A two-component generalization of Burgers equation with Quasi-periodic solutions

Hongfei Pan , Tiecheng Xia *, Dengyuan Chen

Department of Mathematics, Shanghai University, Shanghai 200444, China

Abstract

In this paper, we aim for the theta function representation of quasi-periodic solution and related crucial quantities for a two-component generalization of Burgers equation. Our tools include the theory of algebraic curve, the meromorphic function, Baker-Akhiezer functions, the Dubrovin-type equations for auxiliary divisor, with these tools, the explicit representations for above quantities are obtained.

Keywords: quasi-periodic solutions; theta function; divisor

1 Introduction

The most important physical property of solitons is that they are localized wave packets which survive collisions with other solitons without change of shape. Also, solitons found numerous applications in classical and quantum field theory and in connection with optical communication devices. For a guide to the vast literature on solitons, see for instance [1, 2]. The explicit theta function representations of quasi-periodic solutions (including soliton solutions as special limiting cases) of integrable equations are new approach to construing solutions of integrable nonlinear evolution equations, based on inverse spectral theory and algebro-geometric methods[3, 4, 5, 6]. The construction of all algebro-geometric solutions and their theta function representation of some key hierarchy in $1+1$ -dimensions associated with continuous and discrete models had been done[7, 8]. Based on the nonlinearization technique of Lax pairs and direct method have been proposed by Cao[9], through which algebro-geometric solutions of soliton equations can be obtained[10, 11, 12].

In this paper, we will construct the quasi-periodic solution of the following two-component

* *Corresponding author.* E-mail address: xiatc@shu.edu.cn(T. Xia).

generalization of Burgers equation on the basis of approaches in [13, 14]:

$$\begin{aligned} u_t &= 2u_{xx} + 4u_x v, \\ v_t &= -2v_{xx} - 4uu_x + 4vv_x - 4u_x v, \end{aligned} \quad (1.1)$$

as a reduction case, taking $u = 0$, (1.1) reduces to the burgers equation $v_t = -2v_{xx} + 4vv_x$.

The paper is organized as follows. In Section 2, we introduce the Lenard gradient to derive a hierarchy, in which the second system is our equation (1.1). In section 3, we establish a direct relation between the elliptic variables and the potentials. In section 4, the hyperelliptic Riemann surface of arithmetic genus N and the Abel-Jacobi coordinates are introduced from which the corresponding flows are straightened. Finally, quasi-periodic solution of (1.1) is given in term of the Riemann theta function, according to the asymptotic properties and the algebro-geometric characters of the meromorphic function ϕ , the Baker-Akhiezer function ψ_1 and the hyperelliptic curve \mathcal{K}_N .

2 The hierarchy and Lax pairs of the two-component generalization of Burgers equation

In this section, we shall derive a hierarchy of (1.1). Let's introduce the Lenard gradient sequence $\{S_j\}_{j=0,1,2,\dots}$ by the recursion relation

$$KS_{j-1} = JS_j, \quad j = 1, 2, 3, \dots \quad S_j|_{(u,v)=0} = 0, \quad S_0 = (2, -2u, 1)^T, \quad (2.1)$$

where $S_j = (S_j^{(1)}, S_j^{(2)}, S_j^{(3)})$ and K, J are two operators defined by $(\partial = \partial/\partial x)$:

$$K = \begin{pmatrix} \partial - (u + v) & 0 & 0 \\ 0 & \partial + (u + v) & 0 \\ -u & -1 & \partial \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & -2u \\ -u & -1 & \partial \end{pmatrix}. \quad (2.2)$$

A direct calculation gives from the recursion relation(2.1)that

$$S_1 = (2(u - v), 2u_x + 2u(v - u), 2u)^T, \quad (2.3)$$

$$S_2 = \begin{pmatrix} 2u^2 - 2u_x - 2v_x - 8uv + 2v^2 \\ -2u^3 + 6uu_x + 8u^2v - 2uv^2 - 2u_{xx} - 4u_xv - 2uv_x \\ 2u^2 - 2u_x - 4uv \end{pmatrix}. \quad (2.4)$$

Consider the spectral problem :

$$\psi_x = U\psi, \quad U = \begin{pmatrix} -\frac{1}{2}(\lambda^2 + u + v) & \lambda u \\ -\lambda & \frac{1}{2}(\lambda^2 + u + v) \end{pmatrix}, \quad (2.5)$$

and the auxiliary problem:

$$\psi_{t_m} = V^{(m)}\psi, \quad V^{(m)} = \begin{pmatrix} V_{11}^{(m)} & V_{12}^{(m)} \\ V_{21}^{(m)} & -V_{11}^{(m)} \end{pmatrix}, \quad (2.6)$$

where $V_{11}^{(m)}, V_{12}^{(m)}, V_{21}^{(m)}$ are polynomials of the spectral parameter λ with:

$$\begin{aligned} V_{11}^{(m)} &= \frac{1}{2}S_{mx}^{(1)} + \frac{1}{2}(u+v)S_m^{(1)} + \sum_{j=0}^m S_j^{(3)}\lambda^{2(m-j)+2}, \\ V_{12}^{(m)} &= \sum_{j=0}^m S_j^{(2)}\lambda^{2(m-j)+1}, \\ V_{21}^{(m)} &= \sum_{j=0}^m S_j^{(1)}\lambda^{2(m-j)+1}. \end{aligned} \quad (2.7)$$

Then the compatibility condition of (2.5) and (2.6) yields the zero curvature equation, $U_{t_m} - V_x^{(m)} + [U, V^{(m)}] = 0$, which is equivalent to a hierarchy of nonlinear evolution equations

$$\begin{aligned} u_{t_m} &= S_{mx}^{(2)} + (u+v)S_m^{(2)} + uS_{mx}^{(1)} + u(u+v)S_m^{(1)}, \\ v_{t_m} &= S_{mxx}^{(1)} - \partial(u+v)S_m^{(1)} - S_{mx}^{(2)} - (u+v)S_m^{(2)} - uS_{mx}^{(1)} - u(u+v)S_m^{(1)}. \end{aligned} \quad (2.8)$$

The second equations in (2.8) is ($m = 1$)

$$\begin{aligned} u_{t_1} &= 2u_{xx} + 4u_xv, \\ v_{t_1} &= -2v_{xx} - 4uu_x + 4vv_x - 4u_xv, \end{aligned}$$

and it is our equation (1.1).

3 Evolution of elliptic variables

In this section, we shall establish a relation between the elliptic variables and the potentials. Let $\psi = (\psi_1, \psi_2)^T$ and $\phi = (\phi_1, \phi_2)^T$ be two basic solutions of (2.5) and (2.6), We define a matrix W by

$$W = \frac{1}{2}(\phi\psi^T + \psi\phi^T) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} G & F \\ H & -G \end{pmatrix}. \quad (3.1)$$

It is easy to calculate by (2.5) and (2.6) that

$$W_x = [U, W], \quad W_{t_m} = [V^m, W], \quad (3.2)$$

which implies that $\partial_x \det W = 0$, $\partial_{t_m} \det W = 0$. Equation (3.2) can be written as

$$\begin{aligned} G_x &= \lambda(uH + F), \\ F_x &= -F\lambda^2 - (u+v)F - 2uG\lambda, \\ H_x &= H\lambda^2 + (u+v)H - 2G\lambda, \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} G_{t_m} &= HV_{12}^{(m)} - FV_{21}^{(m)}, \\ F_{t_m} &= 2FV_{11}^{(m)} - 2GV_{12}^{(m)}, \\ H_{t_m} &= 2GV_{21}^{(m)} - 2HV_{11}^{(m)}. \end{aligned} \quad (3.4)$$

We suppose that the functions G, F, H are finite-order polynomials in λ :

$$G = \sum_{j=0}^N g_{2j+1} \lambda^{2(N-j)+1}, \quad F = \sum_{j=0}^N f_{2j} \lambda^{2(N-j)}, \quad H = \sum_{j=0}^N h_{2j} \lambda^{2(N-j)}. \quad (3.5)$$

Substituting (3.5) into (3.3) yields:

$$KG_{j-1} = JG_j (j = 1, 2, \dots, N), \quad JG_0 = 0, \quad KG_N = 0, \quad G_j = (h_{2j}, f_{2j}, g_{2j+1})^T. \quad (3.6)$$

The equation $JG_0 = 0$ has the general solution

$$G_0 = \alpha_0 S_0, \quad (3.7)$$

where α_0 is a constant of integration, and let $\alpha_0 = 1$ without loss of generality. If we take (3.7) as a starting point, then G_j can be recursively determined by the relation (3.6). In fact, noticing $\ker J = \{cS_0 | \forall c \in \mathbb{C}\}$ and acting with the operator $(J^{-1}K)^k$ upon (3.7), we obtain from (3.6) and (2.1) that:

$$G_k = \sum_{j=0}^k \alpha_j S_{k-j}, \quad k = 0, 1, \dots, N, \quad (3.8)$$

where $\alpha_1, \dots, \alpha_k$ are integral constants. The first few members in (3.8) are:

$$\begin{aligned} G_0 &= \begin{pmatrix} 2 \\ -2u \\ 1 \end{pmatrix}, \quad G_1 = \begin{pmatrix} 2(u-v) \\ 2u_x + 2u(v-u) \\ 2u \end{pmatrix} + \alpha_1 \begin{pmatrix} 2 \\ -2u \\ 1 \end{pmatrix}, \\ G_2 &= \begin{pmatrix} 2u^2 - 2u_x - 2v_x - 8uv + 2v^2 \\ -2u^3 + 6uu_x + 8u^2v - 2uv^2 - 2u_{xx} - 4u_xv - 2uv_x \\ 2u^2 - 2u_x - 4uv \end{pmatrix} \\ &\quad + \alpha_1 \begin{pmatrix} 2(u-v) \\ 2u_x + 2u(v-u) \\ 2u \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ -2u \\ 1 \end{pmatrix}. \end{aligned} \quad (3.9)$$

We write F and H as the following finite products:

$$F = -2u \prod_{j=1}^N (\lambda^2 - u_j^2) := -2u \prod_{j=1}^N (\tilde{\lambda} - \tilde{u}_j), \quad H = 2 \prod_{j=1}^N (\lambda^2 - v_j^2) := 2 \prod_{j=1}^N (\tilde{\lambda} - \tilde{v}_j), \quad (3.10)$$

where $\tilde{\lambda} = \lambda^2$, $\tilde{u}_j = u_j^2$, $\tilde{v}_j = v_j^2$, and $\{\tilde{u}_j\}_{j=1, \dots, N}$ and $\{\tilde{v}_j\}_{j=1, \dots, N}$ are called elliptic variables, comparing the coefficients of $\tilde{\lambda}^{N-1}$ in the expressions for F and H in (3.5) and (3.10), respectively, we obtain:

$$\partial \ln u = \sum_{j=1}^N (\tilde{u}_j - \tilde{v}_j), \quad (3.11)$$

$$v - u = \sum_{j=1}^N \tilde{v}_j + \alpha_1. \quad (3.12)$$

Similarly, comparing the coefficients of $\tilde{\lambda}^{N-2}$ and $\tilde{\lambda}^0$ in (3.5) and (3.10), respectively, we have:

$$u^2 - 3u_x - 4uv + v^2 + \frac{u_{xx}}{u} + \frac{2u_x v}{u} + v_x - \alpha_1[\partial \ln u + (v - u)] + \alpha_2 = \sum_{j < k} \tilde{u}_j \tilde{u}_k, \quad (3.13)$$

$$u^2 - u_x - v_x - 4uv + v^2 + \alpha_1(u - v) + \alpha_2 = \sum_{j < k} \tilde{v}_j \tilde{v}_k, \quad (3.14)$$

$$f_{2N} = (-1)^{N+1} 2u \prod_{j=1}^N \tilde{u}_j, \quad h_{2N} = (-1)^N 2 \prod_{j=1}^N \tilde{v}_j. \quad (3.15)$$

Since $\det W$ is a $(2N+1)$ th-order polynomial in $\tilde{\lambda}$ with constant coefficients of the x -flow and t_m -flow, we have:

$$-\det W = G^2 + FH = \prod_{j=1}^{2N+1} (\lambda^2 - \lambda_j^2) = \prod_{j=1}^{2N+1} (\tilde{\lambda} - \tilde{\lambda}_j) = \frac{1}{\tilde{\lambda}} R(\tilde{\lambda}), \quad (3.16)$$

from which we obtain

$$G|_{\tilde{\lambda}=\tilde{u}_k} = \sqrt{\frac{R(\tilde{u}_k)}{\tilde{u}_k}}, \quad G|_{\tilde{\lambda}=\tilde{v}_k} = \sqrt{\frac{R(\tilde{v}_k)}{\tilde{v}_k}}. \quad (3.17)$$

By using (3.3) and (3.10), we get:

$$F_x|_{\tilde{\lambda}=\tilde{u}_k} = 2u \tilde{u}_{k,x} \prod_{j=1, j \neq k}^N (\tilde{u}_k - \tilde{u}_j) = -2u \sqrt{\tilde{u}_k} G|_{\tilde{\lambda}=\tilde{u}_k}, \quad (3.18)$$

$$H_x|_{\tilde{\lambda}=\tilde{v}_k} = -2\tilde{v}_{k,x} \prod_{j=1, j \neq k}^N (\tilde{v}_k - \tilde{v}_j) = -2\sqrt{\tilde{v}_k} G|_{\tilde{\lambda}=\tilde{v}_k}, \quad (3.19)$$

which means:

$$\tilde{u}_{k,x} = \frac{-\sqrt{R(\tilde{u}_k)}}{\prod_{j=1, j \neq k}^N (\tilde{u}_k - \tilde{u}_j)}, \quad v_{k,x} = \frac{\sqrt{R(\tilde{v}_k)}}{\prod_{j=1, j \neq k}^N (\tilde{v}_k - \tilde{v}_j)}, \quad 1 \leq k \leq N. \quad (3.20)$$

In a way similar to the above expression, by using (3.4), (3.10), (3.17), we get the evolution of $\{\tilde{u}_k\}$ and $\{\tilde{v}_k\}$ along the t_m -flow:

$$\tilde{u}_{k,t_m} = -\frac{G|_{\tilde{\lambda}=\tilde{u}_k} V_{12}^{(m)}|_{\tilde{\lambda}=\tilde{u}_k}}{u \prod_{j=1, j \neq k}^N (\tilde{u}_k - \tilde{u}_j)}, \quad \tilde{v}_{k,t_m} = -\frac{G|_{\tilde{\lambda}=\tilde{v}_k} V_{21}^{(m)}|_{\tilde{\lambda}=\tilde{v}_k}}{\prod_{j=1, j \neq k}^N (\tilde{v}_k - \tilde{v}_j)}. \quad (3.21)$$

When $m = 1$, associate with (3.11) and (3.12) and set $t_1 = t$, we have:

$$\tilde{u}_{k,t} = \frac{2\sqrt{R(\tilde{u}_k)}(\tilde{u}_k - \sum_{j=1}^N \tilde{u}_j - \alpha_1)}{\prod_{j=1, j \neq k}^N (\tilde{u}_k - \tilde{u}_j)}, \quad \tilde{v}_{k,t} = \frac{2\sqrt{R(\tilde{v}_k)}(-\tilde{v}_k + \sum_{j=1}^N \tilde{v}_j + \alpha_1)}{\prod_{j=1, j \neq k}^N (\tilde{v}_k - \tilde{v}_j)}. \quad (3.22)$$

4 Quasi-periodic solution

In this section, we shall construct quasi-periodic solution of (1.1). Noticing (3.16), we introduce the hyperelliptic curve \mathcal{K}_N of arithmetic genus N defined by:

$$\mathcal{K}_N : y^2 - R(\tilde{\lambda}) = 0, \quad R(\tilde{\lambda}) = \prod_{j=1}^{2N+2} (\tilde{\lambda} - \tilde{\lambda}_j), \quad \tilde{\lambda}_{2N+2} = 0. \quad (4.1)$$

The curve \mathcal{K}_N can be compactified by joining two points at infinity $P_{\infty\pm}$, $P_{\infty+} = (P_{\infty-})^*$. For notational simplicity the compactification is also denoted by \mathcal{K}_N . Here we assume that the zeros $\tilde{\lambda}_j$, $j = 1, \dots, 2N+2$ of $R(\tilde{\lambda})$ in (4.1) are mutually distinct, then the hyperelliptic curve \mathcal{K}_N becomes nonsingular. According to the definition of \mathcal{K}_N , we can lift the roots $\{\tilde{u}_j\}_{j=1, \dots, N}$, $\{\tilde{v}_j\}_{j=1, \dots, N}$ to \mathcal{K}_N by introducing:

$$\widehat{\tilde{u}}_j(x, t_m) = (\tilde{u}_j(x, t_m), \tilde{\lambda}^{\frac{1}{2}} G(\tilde{u}_j(x, t_m))), \quad j = 1, \dots, N, \quad (4.2)$$

$$\widehat{\tilde{v}}_j(x, t_m) = (\tilde{v}_j(x, t_m), -\tilde{\lambda}^{\frac{1}{2}} G(\tilde{v}_j(x, t_m))), \quad j = 1, \dots, N, \quad (4.3)$$

where $j = 1, \dots, N$, $(x, t_m) \in \mathbb{R}^2$.

We give the definition of the meromorphic function $\phi(\cdot, x, t_m)$ on \mathcal{K}_N by (4.1) and (3.16)

$$\phi(P, x, t_m) = \frac{y + \tilde{\lambda}^{\frac{1}{2}} G}{\tilde{\lambda}^{\frac{1}{2}} F} = \frac{\tilde{\lambda}^{\frac{1}{2}} H}{y - \tilde{\lambda}^{\frac{1}{2}} G}, \quad (4.4)$$

where $P = (\tilde{\lambda}, y) \in \mathcal{K}_N \setminus \{P_{\infty\pm}\}$, hence the divisor of $\phi(\cdot, x, t_m)$ reads[7, 16]

$$(\phi(\cdot, x, t_m)) = \mathcal{D}_{P_{\infty+} \widehat{\underline{v}}(x, t_m)} - \mathcal{D}_{P_{\infty-} \widehat{\underline{u}}(x, t_m)}, \quad (4.5)$$

where

$$\mathcal{D}_{\widehat{\underline{u}}}(x, t_m)(P) = \sum_{j=1}^N \widehat{\tilde{u}}_j(x, t_m), \quad \mathcal{D}_{\widehat{\underline{v}}}(x, t_m)(P) = \sum_{j=1}^N \widehat{\tilde{v}}_j(x, t_m),$$

and $P_{\infty+}$, $\widehat{\tilde{v}}_1(x, t_m), \dots, \widehat{\tilde{v}}_N(x, t_m)$ are the $N+1$ zeros of $\phi(P, x, t_m)$. $P_{\infty-}$, $\widehat{\tilde{u}}_1(x, t_m), \dots, \widehat{\tilde{u}}_N(x, t_m)$ are the $N+1$ poles of $\phi(P, x, t_m)$.

On the basis of the definition of meromorphic function $\phi(\cdot, x, t_m)$ in (4.4), the spectral problem (2.5) and the auxiliary problem (2.6), we can define the Baker-Akhiezer vector $\Psi(\cdot, x, x_0, t_m, t_{m,0})$ on $\mathcal{K}_N \setminus \{P_{\infty+}, P_{\infty-}\}$ by

$$\Psi(P, x, x_0, t_m, t_{m,0}) = \begin{pmatrix} \psi_1(P, x, x_0, t_m, t_{m,0}) \\ \psi_2(P, x, x_0, t_m, t_{m,0}) \end{pmatrix}, \quad (4.6)$$

where

$$\begin{aligned} \psi_1(P, x, x_0, t_m, t_{m,0}) = & \exp\left(\int_{x_0}^x \left(-\frac{1}{2}\right)(\tilde{\lambda} + u(x', t_m) + v(x', t_m)) - \sqrt{\tilde{\lambda}}u(x', t_m)\phi(P, x', t_m)\right)dx' \\ & + \int_{t_{m,0}}^{t_m} (V_{11}^{(m)}(\tilde{\lambda}, x_0, s) - V_{12}^{(m)}(\tilde{\lambda}, x_0, s)\phi(P, x_0, s))ds \end{aligned}, \quad (4.7)$$

$$\psi_2(P, x, x_0, t_m, t_{m,0}) = -\psi_1(P, x, x_0, t_m, t_{m,0})\phi(P, x, t_m), \quad (4.8)$$

with $P \in \mathcal{K}_N \setminus \{P_{\infty+}, P_{\infty-}\}$, $(x, t_m), (x_0, t_{m,0}) \in \mathbb{R}^2$.

Next, we introduce the Riemann surface Γ of the hyperelliptic curve \mathcal{K}_N and equip Γ with a canonical basis of cycles: $a_1, a_2, \dots, a_N; b_1, b_2, \dots, b_N$ which are independent and have intersection numbers as follows:

$$a_i \circ a_j = 0, \quad b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{ij}, \quad i, j = 1, 2, \dots, N.$$

We choose the following set as our basis:

$$\tilde{\omega}_l = \frac{\tilde{\lambda}^{l-1} d\tilde{\lambda}}{\sqrt{R(\tilde{\lambda})}}, \quad l = 1, 2, \dots, N,$$

which are linearly independent homomorphic differentials from each other on Γ , and let

$$A_{ij} = \int_{a_j} \tilde{\omega}_i, \quad B_{ij} = \int_{b_j} \tilde{\omega}_i.$$

It is possible to show that the matrices $A = (A_{ij})$ and $B = (B_{ij})$ are $N \times N$ invertible period matrices [15, 16]. Now we define the matrices C and τ by $C = (C_{ij}) = A^{-1}$, $\tau = (\tau_{ij}) = A^{-1}B$. Then the matrix τ can be shown to symmetric ($\tau_{ij} = \tau_{ji}$) and it has a positive-definite imaginary part ($\text{Im } \tau > 0$). If we normalize $\tilde{\omega}_j$ into the new basis ω_j :

$$\omega_j = \sum_{l=1}^N C_{jl} \tilde{\omega}_l, \quad l = 1, 2, \dots, N,$$

then we have:

$$\begin{aligned} \int_{a_j} \omega_i &= \sum_{l=1}^N C_{jl} \int_{a_j} \tilde{\omega}_l = \sum_{l=1}^N C_{jl} A_{li} = \delta_{ji}, \\ \int_{b_j} \omega_i &= \sum_{l=1}^N C_{jl} \int_{b_j} \tilde{\omega}_l = \sum_{l=1}^N C_{jl} B_{li} = \tau_{ji}. \end{aligned}$$

Now we define the Abel-Jacobi coordinates:

$$\rho_j^{(1)}(x, t_m) = \sum_{k=1}^N \int_{P_0}^{\widehat{u}_k(x, t_m)} \omega_j = \sum_{k=1}^N \sum_{l=1}^N \int_{\lambda(P_0)}^{\widetilde{u}_k(x, t_m)} C_{jl} \frac{\widetilde{\lambda}^{l-1} d\widetilde{\lambda}}{\sqrt{R(\widetilde{\lambda})}}, \quad (4.9)$$

$$\rho_j^{(2)}(x, t_m) = \sum_{k=1}^N \int_{P_0}^{\widehat{v}_k(x, t_m)} \omega_j = \sum_{k=1}^N \sum_{l=1}^N \int_{\lambda(P_0)}^{\widetilde{v}_k(x, t_m)} C_{jl} \frac{\widetilde{\lambda}^{l-1} d\widetilde{\lambda}}{\sqrt{R(\widetilde{\lambda})}}, \quad (4.10)$$

where $\lambda(P_0)$ is the local coordinate of P_0 . From (3.20) and (4.9), we get

$$\partial_x \rho_j^{(1)} = \sum_{k=1}^N \sum_{l=1}^N C_{jl} \frac{\widetilde{u}_k^{l-1} \widetilde{u}_{k,x}}{\sqrt{R(\widetilde{u}_k)}} = - \sum_{k=1}^N \sum_{l=1}^N \frac{C_{jl} \widetilde{u}_k^{l-1}}{\prod_{j=1, j \neq k}^N (\widetilde{u}_k - \widetilde{u}_j)},$$

which implies

$$\partial_x \rho_j^{(1)} = -C_{jN} = \Omega_j^{(1)}, \quad j = 1, 2, \dots, N. \quad (4.11)$$

With the help of the following equality:

$$\sum_{k=1}^N \frac{u_k^{l-1}}{\prod_{i=1, i \neq k}^N (u_k - u_i)} = \begin{cases} \delta_{lN}, & l = 1, 2, \dots, N, \\ \sum_{i_1+i_2+\dots+i_N=l-N} u_1^{i_1} u_2^{i_2} \dots u_N^{i_N}, & l > N. \end{cases}$$

In a similar way, we obtain from (4.9), (4.10), (3.20), (3.21):

$$\partial_t \rho_j^{(1)} = 2C_{j,N-1} - 2\alpha_1 C_{j,N} = \Omega_j^{(2)}, \quad j = 1, 2, \dots, N, \quad (4.12)$$

$$\partial_x \rho_j^{(2)} = -\Omega_j^{(1)}, \quad j = 1, 2, \dots, N, \quad (4.13)$$

$$\partial_t \rho_j^{(2)} = -\Omega_j^{(2)}, \quad j = 1, 2, \dots, N. \quad (4.14)$$

Let \mathcal{T} be the lattice generated by $2N$ vectors δ_j, τ_j , where $\delta_j = (\underbrace{0, \dots, 0}_{j-1}, 1, \underbrace{0, \dots, 0}_{N-j})$ and $\tau_j = \tau \delta_j$, the Jacobian variety of Γ is $\mathcal{J} = \mathbb{C}^N / \mathcal{T}$. On the basis of these results, we obtain the following:

$$\rho_j^{(1)}(x, t) = \Omega_j^{(1)} x + \Omega_j^{(2)} t + \gamma_j^{(1)}, \quad (4.15)$$

$$\rho_j^{(2)}(x, t) = -\Omega_j^{(1)} x - \Omega_j^{(2)} t + \gamma_j^{(2)}, \quad (4.16)$$

where $\gamma_j^{(i)} (i = 1, 2)$ are constants, and

$$\underline{\rho}^{(1)} = (\rho_1^{(1)}, \rho_2^{(1)}, \dots, \rho_N^{(1)})^T, \quad \underline{\rho}^{(2)} = (\rho_1^{(2)}, \rho_2^{(2)}, \dots, \rho_N^{(2)})^T,$$

$$\underline{\Omega}^{(m)} = (\Omega_1^{(m)}, \Omega_2^{(m)}, \dots, \Omega_N^{(m)})^T, \quad \underline{\gamma}^{(m)} = (\gamma_1^{(m)}, \gamma_2^{(m)}, \dots, \gamma_N^{(m)})^T, \quad m = 1, 2.$$

Now we introduce the Abel map $\mathcal{A}(P) : Div(\Gamma) \rightarrow \mathcal{J}$:

$$\mathcal{A}(P) = \int_{P_0}^P \underline{\omega}, \quad \underline{\omega} = (\omega_1, \omega_2, \dots, \omega_N)^T,$$

$$\mathcal{A}\left(\sum_k n_k P_k\right) = \sum_k n_k \mathcal{A}(P_k), \quad P, P_k \in \mathcal{K}_N,$$

the Riemann theta function is defined as [7, 15, 16]

$$\begin{aligned} \theta(P, D_{\underline{u}(x, t_m)}) &= \theta(\underline{\Lambda} - \mathcal{A}(P) + \underline{\rho}^{(1)}), \\ \theta(P, D_{\underline{v}(x, t_m)}) &= \theta(\underline{\Lambda} - \mathcal{A}(P) + \underline{\rho}^{(2)}), \end{aligned} \quad (4.17)$$

where $\underline{\Lambda} = (\Lambda_1, \dots, \Lambda_N)$ is defined by:

$$\Lambda_j = \frac{1}{2}(1 + \tau_{jj}) - \sum_{i=1, i \neq j}^N \int_{a_i} \omega_i \int_{Q_0}^P \omega_j, \quad j = 1, \dots, N.$$

In order to derive the quasi-periodic solution of (1.1), now we turn to the asymptotic properties of the meromorphic function ϕ and Baker-Akhiezer function ψ_1 .

Lemma 4.1. Suppose that $u(x, t_m), v(x, t_m) \in C^\infty(\mathbb{R}^2)$ satisfy the equation (1.1). Moreover, let $P = (\tilde{\lambda}, y) \in \mathcal{K}_N \setminus \{P_{\infty\pm}, P_0\}$, $(x, x_0) \in \mathbb{R}^2$. Then:

$$\phi(P) \underset{\zeta \rightarrow 0}{=} \begin{cases} -\zeta^{\frac{1}{2}} - v\zeta^{\frac{3}{2}} + O(\zeta^{\frac{5}{2}}) & \text{as } P \rightarrow P_{\infty+}, \\ -\frac{1}{u}\zeta^{-\frac{1}{2}} - \left(\frac{u_x}{u^2} + \frac{v}{u}\right)\zeta^{\frac{1}{2}} + O(\zeta^{\frac{3}{2}}) & \text{as } P \rightarrow P_{\infty-}, \end{cases} \quad (4.18)$$

$$\phi(P) \underset{\zeta \rightarrow 0}{=} 1 - 2\partial_t^{-1}((u+v)(u-v) + (u_x - v_x)) + O(\zeta) \quad \text{as } P \rightarrow P_0, \quad (4.19)$$

$$\psi_1(P, x, x_0, t_m, t_{m,0}) \underset{\zeta \rightarrow 0}{=} \begin{cases} \exp(-\frac{1}{2}\zeta^{-1}(x - x_0) + \zeta^{-m-1}(t_m - t_{m,0}) + O(1)) & \text{as } P \rightarrow P_{\infty+}, \\ \exp(\frac{1}{2}\zeta^{-1}(x - x_0) - \zeta^{-m-1}(t_m - t_{m,0}) + O(1)) & \text{as } P \rightarrow P_{\infty-}, \end{cases} \quad (4.20)$$

Proof. We first prove ϕ satisfies the Riccati-type equations:

$$\phi_x(P) - [\tilde{\lambda} + (u+v)]\phi(P) - \sqrt{\tilde{\lambda}}u\phi^2(P) = \sqrt{\tilde{\lambda}}, \quad (4.21)$$

$$\phi_t(P) + 2V_{11}^{(1)}\phi(P) - V_{12}^{(1)}\phi^2(P) = -V_{21}^{(1)}. \quad (4.22)$$

The local coordinates $\zeta = \tilde{\lambda}^{-1}$ near $P_{\infty\pm}$ and $\zeta = \tilde{\lambda}$ near P_0 , from (4.4), (3.3), (4.1), we have

$$\phi_x = \frac{\tilde{\lambda}G_x F - (y + \tilde{\lambda}^{\frac{1}{2}}G)\tilde{\lambda}^{\frac{1}{2}}F_x}{\tilde{\lambda}F^2} = \tilde{\lambda}^{\frac{1}{2}} + [\tilde{\lambda} + (u+v)]\phi + \frac{(uH + 2uG\phi)\tilde{\lambda}^{\frac{1}{2}}}{F}, \quad (4.23)$$

$$\phi^2 = \frac{y^2 + 2\tilde{\lambda}^{\frac{1}{2}}yG + \tilde{\lambda}G^2}{\tilde{\lambda}F^2} = \frac{2\tilde{\lambda}G^2 + \tilde{\lambda}FH + 2\tilde{\lambda}^{\frac{1}{2}}yG}{\tilde{\lambda}F^2} = \frac{2G\phi + H}{F}, \quad (4.24)$$

according to (4.23) and (4.24), we have (4.21). Similarly, by using (4.4), (3.4), (4.1), we have (4.22). And then, inserting the ansatz $\phi = \phi_1\tilde{\lambda}^{-\frac{1}{2}} + \phi_2\tilde{\lambda}^{-\frac{3}{2}} + O(\tilde{\lambda}^{-\frac{5}{2}})$ into (4.21), we get the first line of (4.18). Inserting the ansatz $\phi = \phi_{-1}\tilde{\lambda}^{\frac{1}{2}} + \phi_1\tilde{\lambda}^{-\frac{1}{2}} + O(\tilde{\lambda}^{-\frac{3}{2}})$ into (4.21), we get the second line of (4.18). In exactly the same manner, inserting the ansatz $\phi = \phi_0 + \phi_1\tilde{\lambda} + O(\tilde{\lambda}^2)$ into (4.22) immediately yields (4.19).

In the following, we will prove (4.20). From (4.7) and (4.18):

$$\begin{aligned} & \exp(\int_{x_0}^x ((-\frac{1}{2})(\tilde{\lambda} + u(x', t_m) + v(x', t_m)) - \tilde{\lambda}^{\frac{1}{2}}u(x', t_m)\phi(P, x', t_m))dx') \\ &= \exp(\int_{x_0}^x ((-\frac{1}{2})(\zeta^{-1} + u + v) - \zeta^{-\frac{1}{2}}u\phi)dx') \\ &\stackrel{=}{\underset{\zeta \rightarrow 0}{\left\{ \begin{aligned} & \exp(\int_{x_0}^x ((-\frac{1}{2})(\zeta^{-1} + u + v) - \zeta^{-\frac{1}{2}}u(-\zeta^{\frac{1}{2}} - v\zeta^{\frac{3}{2}} + O(\zeta^{\frac{5}{2}}))) \text{ as } P \rightarrow P_{\infty+}, \\ & \exp(\int_{x_0}^x ((-\frac{1}{2})(\zeta^{-1} + u + v) - \zeta^{-\frac{1}{2}}u(-\frac{1}{u}\zeta^{-\frac{1}{2}} - (\frac{u_x}{u^2} + \frac{v}{u})\zeta^{\frac{1}{2}} + O(\zeta^{\frac{3}{2}}))) \text{ as } P \rightarrow P_{\infty-}, \\ & \exp(-\frac{1}{2}\zeta^{-1}(x - x_0) + O(1)) \text{ as } P \rightarrow P_{\infty+}, \\ & \exp(\frac{1}{2}\zeta^{-1}(x - x_0) + O(1)) \text{ as } P \rightarrow P_{\infty-}. \end{aligned} \right.}} \end{aligned} \quad (4.25)$$

From (4.1) and (3.16), we have

$$\begin{aligned} y &= \mp \sqrt{R(\tilde{\lambda})} \\ &= \mp \tilde{\lambda}^{\frac{1}{2}} \prod_{j=1}^{2N+1} (\tilde{\lambda} - \tilde{\lambda}_j)^{\frac{1}{2}} \\ &= \mp \zeta^{-N-1} \prod_{j=1}^{2N+1} (1 - \tilde{\lambda}_j \zeta)^{\frac{1}{2}} \\ &\stackrel{=}{\underset{\zeta \rightarrow 0}{\mp \zeta^{-N-1} \prod_{j=1}^{2N+1} (1 - \frac{1}{2}\tilde{\lambda}_j \zeta - \frac{1}{8}\tilde{\lambda}_j^2 \zeta^2 + O(\zeta^3))}} \\ &\stackrel{=}{\underset{\zeta \rightarrow 0}{\mp \zeta^{-N-1} (1 + \epsilon_1 \zeta + \epsilon_2 \zeta^2 + O(\zeta^3))}} \text{ as } P \rightarrow P_{\infty\pm}, \end{aligned} \quad (4.26)$$

where $\epsilon_1 = -\frac{1}{2} \sum_{j=1}^{2N+1} \tilde{\lambda}_j$, $\epsilon_2 = \frac{1}{2} \sum_{j < k} \tilde{\lambda}_j \tilde{\lambda}_k - \frac{1}{8} (\sum_{j=1}^{2N+1} \tilde{\lambda}_j)^2$. From (3.10) and (2.7), we can derive

$$\begin{aligned} (\tilde{\lambda}^{\frac{1}{2}}F)^{-1} &= \tilde{\lambda}^{-\frac{1}{2}} \frac{1}{-2u} \prod_{j=1}^N \frac{1}{\tilde{\lambda} - \tilde{u}_j} \\ &= -\frac{1}{2u} \zeta^{N+\frac{1}{2}} \prod_{j=1}^N \frac{1}{1 - \tilde{u}_j \zeta} \\ &\stackrel{=}{\underset{\zeta \rightarrow 0}{-\frac{1}{2u} \zeta^{N+\frac{1}{2}} (1 + \sum_{j=1}^N \tilde{u}_j \zeta + O(\zeta^2))}} \text{ as } P \rightarrow P_{\infty\pm}, \end{aligned} \quad (4.27)$$

$$\begin{aligned} V_{12}^{(m)} &= \sum_{j=0}^m S_j^{(2)} \tilde{\lambda}^{m-j+\frac{1}{2}} \\ &= \zeta^{-\frac{1}{2}} (S_0^{(2)} \zeta^{-m} + S_1^{(2)} \zeta^{-m+1} + \dots + S_{m-1}^{(2)} \zeta^{-1} + S_m^{(2)}) \end{aligned} \quad (4.28)$$

combining (4.4), (3.4), we have:

$$\begin{aligned}
& \exp(\int_{t_{m,0}}^{t_m} (V_{11}^{(m)}(\tilde{\lambda}, x_0, s) - V_{12}^{(m)}(\tilde{\lambda}, x_0, s))\phi(P, x_0, s))ds \\
&= \exp(\int_{t_{m,0}}^{t_m} (V_{11}^{(m)} - V_{12}^{(m)} \frac{y+\tilde{\lambda}^{\frac{1}{2}}G}{\tilde{\lambda}^{\frac{1}{2}}F})ds) \\
&= \exp(\int_{t_{m,0}}^{t_m} (\frac{-y}{\tilde{\lambda}^{\frac{1}{2}}F} V_{12}^{(m)} + \frac{1}{2} \frac{F_{t_m}}{F})ds) \\
&\stackrel{\zeta \rightarrow 0}{=} \exp(\int_{t_{m,0}}^{t_m} (\pm \zeta^{-N-1} (1 + O(\zeta)) \frac{\zeta^{-\frac{1}{2}}(S_0^{(2)}\zeta^{-m} + S_1^{(2)}\zeta^{-m+1} + \dots + S_{m-1}^{(2)}\zeta^{-1} + S_m^{(2)})}{\zeta^{-N-\frac{1}{2}}(f_0 + f_2\zeta + \dots + f_{2N}\zeta^N)} + \frac{u_{t_m}(x_0, s)}{2u(x_0, s)} + O(\zeta))ds) \\
&\stackrel{\zeta \rightarrow 0}{=} \exp(\int_{t_{m,0}}^{t_m} (\pm \zeta^{-m-1} + O(\zeta) + \frac{u_{t_m}(x_0, s)}{2u(x_0, s)})ds) \\
&\stackrel{\zeta \rightarrow 0}{=} (\frac{u(x_0, t_m)}{u(x_0, t_{m,0})})^{\frac{1}{2}} \exp(\pm \zeta^{-m-1}(t_m - t_{m,0}) + O(\zeta)) \quad \text{as } P \rightarrow P_{\infty \pm}, \\
&\stackrel{\zeta \rightarrow 0}{=} \begin{cases} \exp(\zeta^{-m-1}(t_m - t_{m,0}) + O(1)) & \text{as } P \rightarrow P_{\infty +}, \\ \exp(-\zeta^{-m-1}(t_m - t_{m,0}) + O(1)) & \text{as } P \rightarrow P_{\infty -}, \end{cases}
\end{aligned} \tag{4.29}$$

according to the definition of ψ_1 in (4.7), (4.25) and (4.29), we can obtain (4.20). \square

Next, we shall derive the representation of $\phi, \psi_1, \psi_2, u(x, t_m), v(x, t_m)$ in term of the Riemann theta function. Let $\omega_{P_{\infty+}, P_{\infty-}}^{(3)}$ be the normalized differential of the third kind holomorphic on $\mathcal{K}_N \setminus \{P_{\infty+}, P_{\infty-}\}$ with simples at $P_{\infty+}$ and $P_{\infty-}$ and residues 1 and -1 respectively,

$$\omega_{P_{\infty+}, P_{\infty-}}^{(3)} = \frac{1}{y} \prod_{j=1}^N (\tilde{\lambda} - \lambda_j) d\tilde{\lambda} \stackrel{\zeta \rightarrow 0}{=} (\pm \frac{1}{2} \zeta^{-1} + O(1)) d\zeta \quad \text{as } P \rightarrow P_{\infty \pm}, \tag{4.30}$$

here the constants $\{\lambda_j | \lambda_j \in \mathbb{C}, j = 1, \dots, N\}$ are uniquely determined by the normalization

$$\int_{a_j} \omega_{P_{\infty+}, P_{\infty-}}^{(3)} = 0, \quad j = 1, \dots, N, \tag{4.31}$$

and ζ in (4.30) denotes the local coordinate $\zeta = \tilde{\lambda}^{-1}$ for P near $P_{\infty \pm}$. Moreover,

$$\int_{Q_0}^P \omega_{P_{\infty+}, P_{\infty-}}^{(3)} \stackrel{\zeta \rightarrow 0}{=} \pm (\frac{1}{2} \ln(\zeta) - \ln \omega_0 + O(\zeta)) \quad \text{as } P \rightarrow P_{\infty \pm}, \tag{4.32}$$

and

$$\int_{Q_0}^P \omega_{P_{\infty+}, P_{\infty-}}^{(3)} \stackrel{\zeta \rightarrow 0}{=} \ln \omega_0 + O(\zeta) \quad \text{as } P \rightarrow P_0, \tag{4.33}$$

and ζ in (4.33) denotes the local coordinate $\zeta = \tilde{\lambda}$ for P near P_0 .

Let $\omega_{P_{\infty \pm}, r}^{(2)}$, $r \in \mathbb{N}_0$, be normalized differentials of the second kind with a unique pole at $P_{\infty \pm}$, and principal part is $\zeta^{-2-r} d\zeta$ near $P_{\infty \pm}$, satisfying

$$\int_{a_j} \omega_{P_{\infty \pm}, r}^{(2)} = 0, \quad j = 1, \dots, N,$$

then we can define $\Omega_0^{(2)}$ and $\Omega_{m-1}^{(2)}$ by

$$\Omega_0^{(2)} = \omega_{P_{\infty+}, 0}^{(2)} - \omega_{P_{\infty-}, 0}^{(2)}, \tag{4.34}$$

$$\Omega_{m-1}^{(2)} = \sum_{l=0}^{m-1} \alpha_{m-1-l} (l+1) (\omega_{P_{\infty+},l}^{(2)} - \omega_{P_{\infty-},l}^{(2)}), \quad (4.35)$$

where α_{m-1-l} , $j = 0, \dots, m-1$ are the integral constants in (3.8), so we have

$$\int_{a_j} \Omega_0^{(2)} = 0, \int_{a_j} \Omega_{m-1}^{(2)}, \quad j = 1, \dots, N, \quad (4.36)$$

$$\int_{Q_0}^P \Omega_0^{(2)} \underset{\zeta \rightarrow 0}{=} \mp \left(\frac{1}{2} \zeta^{-1} + e_{0,0} + O(\zeta) \right) \text{ as } P \rightarrow P_{\infty\pm}, \quad (4.37)$$

$$\int_{Q_0}^P \Omega_{m-1}^{(2)} \underset{\zeta \rightarrow 0}{=} \mp \left(\sum_{l=0}^{m-1} \alpha_{m-1-l} \zeta^{-2-l} + e_{m-1,0} + O(\zeta) \right) \text{ as } P \rightarrow P_{\infty\pm} \quad (4.38),$$

for some constants $e_{0,0}, e_{m-1,0} \in \mathbb{C}$.

If $D_{\widehat{\underline{u}}(x,t_m)}$ or $D_{\widehat{\underline{v}}(x,t_m)}$ in (4.5) is assumed to be nonspecial[7], then according to Riemann's theorem[7, 16], the definition and asymptotic properties of the meromorphic function $\phi(P, x, t_m)$, $\phi(P, x, t_m)$ has expressions of the following type:

$$\phi(P, x, t_m) = C(x, t_m) \frac{\theta(P, D_{\widehat{\underline{v}}(x,t_m)})}{\theta(P, D_{\widehat{\underline{u}}(x,t_m)})} \exp\left(\int_{Q_0}^P \omega_{P_{\infty+}, P_{\infty-}}^{(3)}\right), \quad (4.39)$$

where $C(x, t_m)$ is independent of $P \in \mathcal{K}_N$.

Theorem 4.1. Let $P = (\tilde{\lambda}, y) \in \mathcal{K}_N \setminus P_{\infty\pm}, P_0$, $(x, t_m), (x_0, t_{m,0}) \in \Omega$, where $\Omega \subseteq \mathbb{R}^2$ is open and connected. Suppose $u(\cdot, t_m), v(\cdot, t_m) \in C^\infty(\Omega)$, $u(x, \cdot), v(x, \cdot) \in C^1(\Omega)$, $x \in \mathbb{R}, t_m \in \mathbb{R}$, satisfy the equation (1.1), and assume that $\tilde{\lambda}_j, 1 \leq j \leq 2N+2$ ($\tilde{\lambda}_{2N+2} = 0$) in (3.16) satisfy $\tilde{\lambda}_j \in \mathbb{C}$ and $\tilde{\lambda}_j \neq \tilde{\lambda}_k$ for $j \neq k$. Moreover, suppose that $D_{\widehat{\underline{u}}}$ or equivalently, $D_{\widehat{\underline{v}}}$, is nonspecial for $(x, t_m) \in \Omega$. Then

$$\phi(P, x, t_m) = -\omega_0 \frac{\theta(P_{\infty+}, D_{\widehat{\underline{u}}(x,t_m)}) \theta(P, D_{\widehat{\underline{v}}(x,t_m)})}{\theta(P_{\infty+}, D_{\widehat{\underline{v}}(x,t_m)}) \theta(P, D_{\widehat{\underline{u}}(x,t_m)})} \exp\left(\int_{Q_0}^P \omega_{P_{\infty+}, P_{\infty-}}^{(3)}\right), \quad (4.40)$$

$$\begin{aligned} \psi_1(P, x, x_0, t_m, t_{m,0}) &= \frac{\theta(P_{\infty+}, D_{\widehat{\underline{u}}(x_0,t_{m,0})}) \theta(P, D_{\widehat{\underline{u}}(x,t_m)})}{\theta(P_{\infty+}, D_{\widehat{\underline{u}}(x,t_m)}) \theta(P, D_{\widehat{\underline{u}}(x_0,t_{m,0})})} \\ &\times \exp\left(\left(\int_{Q_0}^P \Omega_0^{(2)} + e_{0,0}\right)(x - x_0) + \left(\int_{Q_0}^P \Omega_{m-1}^{(2)} + e_{m-1,0}\right)(t_m - t_{m,0})\right), \end{aligned} \quad (4.41)$$

$$\begin{aligned} \psi_2(P, x, x_0, t_m, t_{m,0}) &= \omega_0 \frac{\theta(P_{\infty+}, D_{\widehat{\underline{u}}(x,t_m)}) \theta(P, D_{\widehat{\underline{v}}(x,t_m)}) \theta(P_{\infty+}, D_{\widehat{\underline{u}}(x_0,t_{m,0})})}{\theta(P_{\infty+}, D_{\widehat{\underline{u}}(x,t_m)}) \theta(P, D_{\widehat{\underline{u}}(x_0,t_{m,0})}) \theta(P_{\infty+}, D_{\widehat{\underline{u}}(x,t_m)})} \\ &\times \exp\left(\left(\int_{Q_0}^P \Omega_0^{(2)} + e_{0,0}\right)(x - x_0) + \left(\int_{Q_0}^P \Omega_{m-1}^{(2)} + e_{m-1,0}\right)(t_m - t_{m,0})\right) \\ &\times \exp\left(\int_{Q_0}^P \omega_{P_{\infty+}, P_{\infty-}}^{(3)}\right) \end{aligned} \quad (4.42)$$

finally, $u(x, t_m)$ is of the form

$$u(x, t_m) = \frac{1}{\omega_0^2} \frac{\theta(P_{\infty-}, D_{\widehat{\underline{u}}(x,t_m)}) \theta(P_{\infty+}, D_{\widehat{\underline{v}}(x,t_m)})}{\theta(P_{\infty-}, D_{\widehat{\underline{v}}(x,t_m)}) \theta(P_{\infty+}, D_{\widehat{\underline{u}}(x,t_m)})}, \quad (4.43)$$

and $v(x, t_m)$ is determined by

$$u(x, t_m) + v(x, t_m) + u_x(x, t_m) - v_x(x, t_m) = \frac{\omega_0}{2} \partial_t \frac{\theta(P_{\infty+}, D_{\hat{\underline{u}}(x, t_m)}) \theta(P_0, D_{\hat{\underline{v}}(x, t_m)})}{\theta(P_{\infty+}, D_{\hat{\underline{v}}(x, t_m)}) \theta(P_0, D_{\hat{\underline{u}}(x, t_m)})}. \quad (4.44)$$

proof. We start with the proof of the theta function representation (4.41). Without loss of generality, it suffices to treat the special case of (3.8) when $\alpha_0 = 1, \alpha_k = 0, 1 \leq k \leq N$. First, we assume

$$\tilde{u}_j(x, t_m) \neq \tilde{u}_k(x, t_m), \text{ for } j \neq k, \text{ and } (x, t_m) \in \tilde{\Omega}, \quad (4.45)$$

for appropriate $\tilde{\Omega} \subseteq \Omega$, and define the right-hand side of (4.41) to be $\tilde{\psi}_1$. In order to prove $\psi = \tilde{\psi}$, we investigate the local zeros and poles of ψ_1 . From (3.10), (3.20), (3.21), (4.4), we have

$$\begin{aligned} \sqrt{\lambda} u(x', t_m) \phi(P, x', t_m) & \underset{P \rightarrow \hat{\tilde{u}}_j(x', t_m)}{=} \sqrt{\lambda} u(x', t_m) \frac{2y(\hat{\tilde{u}}_j(x', t_m))}{\sqrt{\lambda}(-2u) \prod_{k=1, k \neq j} (\tilde{u}_j(x', t_m) - \tilde{u}_k(x', t_m))} \frac{1}{\lambda - \tilde{u}_j(x', t_m)} \\ & \underset{P \rightarrow \hat{\tilde{u}}_j(x', t_m)}{=} \frac{\tilde{u}_{j,x'}}{\lambda - \tilde{u}_j(x', t_m)} \\ & \underset{P \rightarrow \hat{\tilde{u}}_j(x', t_m)}{=} -\partial_{x'} \ln(\lambda - \tilde{u}_j(x', t_m)) + O(1). \end{aligned} \quad (4.46)$$

And similarly

$$V_{12}^{(m)}(\tilde{\lambda}, x_0, s) \phi(P, x_0, s) \underset{P \rightarrow \hat{\tilde{u}}_j(x_0, s)}{=} -\partial_s \ln(\lambda - \tilde{u}_j(x_0, s)) + O(1), \quad (4.47)$$

then (4.46) and (4.47) together with (4.7) yields

$$\psi_1(P, x, x_0, t_m, t_{m,0}) = \begin{cases} (\tilde{\lambda} - \tilde{u}_j(x, t_m)) O(1) & \text{as } P \rightarrow \hat{\tilde{u}}_j(x, t_m) \neq \hat{\tilde{u}}_j(x_0, t_{m,0}), \\ O(1) & \text{as } P \rightarrow \hat{\tilde{u}}_j(x, t_m) = \hat{\tilde{u}}_j(x_0, t_{m,0}), \\ (\tilde{\lambda} - \tilde{u}_j(x_0, t_{m,0}))^{-1} O(1) & \text{as } P \rightarrow \hat{\tilde{u}}_j(x_0, t_{m,0}) \neq \hat{\tilde{u}}_j(x, t_m), \end{cases} \quad (4.48)$$

where $P = (\tilde{\lambda}, t_m) \in \mathcal{K}_N, (x, t_m), (x_0, t_{m,0}) \in \tilde{\Omega}$ and $O(1) \neq 0$. Hence ψ_1 and $\tilde{\psi}_1$ have identical zeros and poles on $\mathcal{K}_N \setminus \{P_{\infty\pm}\}$, which are all simple by hypothesis (4.45). It remains to study the behavior of ψ_1 and $\tilde{\psi}_1$ near $P_{\infty\pm}$, by (4.20), (4.37), (4.38), (4.41), we can easy find that ψ_1 and $\tilde{\psi}_1$ share the same singularities and zeros, and the Riemann-Roch-type uniqueness[7] proves that $\psi_1 = \tilde{\psi}_1$, hence (4.41) holds subject to (4.45).

Substituting (4.32), (4.33) into (4.39) and comparing with (4.18) and (4.19), we obtain

$$u(x, t_m) = -\frac{1}{C(x, t_m) \omega_0} \frac{\theta(P_{\infty-}, D_{\hat{\underline{u}}(x, t_m)})}{\theta(P_{\infty-}, D_{\hat{\underline{v}}(x, t_m)})}, \quad C(x, t_m) = -\omega_0 \frac{\theta(P_{\infty+}, D_{\hat{\underline{u}}(x, t_m)})}{\theta(P_{\infty+}, D_{\hat{\underline{v}}(x, t_m)})}, \quad (4.49)$$

according to (4.49), we have (4.43) and (4.40), and ψ_2 in (4.42) from $\psi_2 = -\phi\psi_1$. Substituting (4.33) into (4.39) and comparing with (4.19), together with $C(x, t_m)$ in (4.49), we get (4.44).

Hence, we prove this theorem on $\tilde{\Omega}$. The extension of all these results from $\tilde{\Omega}$ to Ω follows by continuity of the Abel map and the nonspecial nature of $D_{\hat{u}}$ or $D_{\hat{v}}$ on Ω . \square

Therefore, the quasi-periodic solution of (1.1) are (4.43) and (4.44) for $m = 1$.

5 Acknowledgements

The Project is in part supported by the Natural Science Foundation of China (Grant No. 11271008), the First-class Discipline of University in Shanghai and the Shanghai Univ. Leading Academic Discipline Project (A.13-0101-12-004).

References

- [1] S. Novikov, S. V. Manakov, L. P. Pitaevskii and V. E. Zakharov, Theory of solitons: The Inverse Scattering Method, Consultants Bureau, New York, 1984.
- [2] A. Degasperis, Solitons, American Journal of Physics, 1998; **66**: 486-497.
- [3] E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its, and V. B. Matveev, Algebro-Geometric Approach to Nonlinear Integrable Equations, Springer, Berlin, 1994.
- [4] B. A. Dubrovin, Theta functions and nonlinear equations, Russian Mathematical Surveys, 1981; **36**: 11-80.
- [5] A. R. Its and V. B. Matveev, Schrodinger operators with finite-gap spectrum and N-soliton solutions of the Korteweg-de Vries equation, Theoretical and Mathematical Physics, 1975; **23**: 343-355.
- [6] D. P. Novikov, Algebraic-geometric solutions of the Harry Dym equation, Siberian Mathematical Journal, 1999; **40**: 136-140.
- [7] F. Gesztesy and H. Holden, Soliton Equations and their Algebro-Geometric Solutions(Volnme I), Cambridge University Press, Cambridge, 2003.
- [8] F. Gesztesy and H. Holden, J. Michor, G. Teschl, Soliton Equations and their Algebro-Geometric Solutions(Volnme II), Cambridge University Press, Cambridge, 2008.
- [9] C. W. Cao, Nonlinearization of the Lax System for AKNS Hierarchy, Science in China (Series A)1990; **33**:528-536.
- [10] C. W. Cao, Y. T. Wu and X. G. Geng, On quasi-periodic solutions of the 2+1 dimensional Caudrey-Dodd-Gibbon-Kotera-Sawada equation, Physics Letters A, 1999; **256**: 59-65.

- [11] C. W. Cao, X. G. Geng and H. Y. Wang, Algebro-geometric solution of the 2+1 dimensional Burgers equation with a discrete variable, *Journal of Mathematical Physics*, 2002; **43**: 621-643.
- [12] X. G. geng and H. H. Dai, Algebro-geometric solutions of $(2 + 1)$ -dimensional coupled modified Kadomtsev-Petviashvili equations, *Journal of Mathematical Physics*, 2000; **41**: 337-348.
- [13] Y. Hou, E. G. Fan and P. Zhao, Algebro-geometric solutions for the Gerdjikov-Ivanov hierarchy, *Journal of Mathematical Physics*, 2013; **54**: 073505.
- [14] Bo Xue, Fang Li and Xianguo Geng, Quasi-periodic solutions of Coupled KDV Type Equations, *Journal of Nonlinear Mathematical Physics*, 2013; **20**: 61-77.
- [15] C. L. Siegel, *Topics in Complex Function Theory*. **2**, Wiley, New York, 1971.
- [16] P. Griffiths and J. Harris, *Principles of Algebraic geometry*, Wiley, New York, 1994.